

Poisson Actions up to Homotopy and their Quantization

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Abstract. Symmetries of Poisson manifolds are in general quantized just to symmetries up to homotopy of the quantized algebra of functions. It is therefore interesting to study symmetries up to homotopy of Poisson manifolds. We notice that they are equivalent to Poisson principal bundles and describe their quantization to symmetries up to homotopy of the quantized algebras of functions, using the formality theorem of Kontsevich.

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0. Introduction

An action of a Lie algebra \mathfrak{g} on a Poisson manifold M is Poisson, if it preserves the Poisson structure on M . In general it is not possible to quantize M so that \mathfrak{g} would act by derivations on the quantized algebra of functions. On the other hand, as a simple application of the formality theorem of Kontsevich [3], a Poisson action up to homotopy (a generalization of Poisson actions, defined below) is easily quantized to an action up to homotopy on the quantized algebra of functions. It is therefore interesting to study the geometry of Poisson actions up to homotopy, together with its generalizations where \mathfrak{g} is replaced by a Lie bialgebra. Namely, it turns out that an up to homotopy Poisson action is the same as a Poisson principal bundle. This geometry sheds a new light even on some pretty standard things, e.g. relations between Poisson and Hamiltonian actions.

1. Up to Homotopy Poisson Actions

An *up to homotopy Poisson action* of a Lie algebra \mathfrak{g} (\mathfrak{g} -HPA for short) on a Poisson manifold M is an extension \mathfrak{g}_M of \mathfrak{g} by the Lie algebra $C^\infty(M)$ (with the

Poisson bracket used as the Lie bracket), such that for every $X \in \mathfrak{g}_M$, the map $C^\infty(M) \rightarrow C^\infty(M)$, $f \mapsto [X, f]$, is a derivation (i.e. a vector field on M). A *morphism* from a \mathfrak{g} -HPA on a Poisson manifold M_1 to a \mathfrak{g} -HPA on M_2 is a Poisson map $f: M_1 \rightarrow M_2$ and a morphism of the extensions of \mathfrak{g} going the opposite direction, equal to $f^*: C^\infty(M_2) \rightarrow C^\infty(M_1)$ on the kernels.

A \mathfrak{g} -HPA on a point is a central extension of \mathfrak{g} by \mathbb{R} . Given a true Poisson action of \mathfrak{g} on a Poisson manifold M , the corresponding extension is the semidirect product. Any splitting of a \mathfrak{g} -HPA to a semidirect product turns it into a true Poisson action (\mathfrak{g}_M has a Poisson action on M by definition, a splitting $\mathfrak{g} \rightarrow \mathfrak{g}_M$ gives us then a Poisson action of \mathfrak{g}). In other words, an ordinary Poisson action on M is the same as a \mathfrak{g} -HPA \mathfrak{g}_M on M and a morphism from \mathfrak{g}_M to the trivial \mathfrak{g} -HPA on a point.

Let us give an equivalent definition using the Maurer–Cartan equation. If M is a manifold, let L_M denote the differential graded Lie algebra (DGLA) of multivector fields $\Gamma(\wedge TM)[1]$ (with Schouten bracket and zero differential). A \mathfrak{g} -HPA on M is a degree-1 element σ of the DGLA $\wedge \mathfrak{g}^* \otimes L_M$, satisfying the Maurer–Cartan (MC) equation

$$d\sigma + [\sigma, \sigma]/2 = 0. \quad (1)$$

As we'll see easily, this is indeed equivalent to the previous definition, when we choose a splitting of the extension $\mathfrak{g}_M = \mathfrak{g} \oplus C^\infty(M)$ (direct sum of vector spaces – not direct (or semidirect) sum of Lie algebras!).

Let us decompose σ into bihomogeneous parts

$$\sigma = \sigma^0 + \sigma^1 + \sigma^2 \quad (2)$$

(the superscript is the degree in $\wedge \mathfrak{g}^*$). Then σ^0 is a Poisson structure on M , and we get a Lie bracket on $\mathfrak{g}_M = \mathfrak{g} \oplus C^\infty(M)$ via $[u, v] = [u, v]_{\mathfrak{g}} + \sigma^2(u, v)$, $[u, f] = \mathcal{L}_{\sigma^1(u)}f$, $[f, g] = \{f, g\}$ (for any $u, v \in \mathfrak{g}$ and $f, g \in C^\infty(M)$).

Yet another equivalent formulation – we have an L_∞ -morphism from \mathfrak{g} to the DGLA L_{M, σ^0} (where L_{M, σ^0} is the same as L_M , but with differential equal to ad_{σ^0}).

Just for the fun of it, let us plug the decomposition (2) into the MC equation (1) and write all the results explicitly:

1. $[\sigma^0, \sigma^0] = 0$, i.e. σ^0 is a Poisson structure on M
2. $[\sigma^0, \sigma^1] = 0$, i.e. $\sigma^1 \in \mathfrak{g}^* \otimes \Gamma(TM)$ maps elements of \mathfrak{g} to vector fields preserving the Poisson structure σ^0
3. $d\sigma^1 + [\sigma^1, \sigma^1]/2 + [\sigma^0, \sigma^2] = 0$, i.e. σ^1 is not necessarily an action of \mathfrak{g} on M , but rather for any $u, v \in \mathfrak{g}$ we have

$$[\sigma^1(u), \sigma^1(v)] = \sigma^1([u, v]) + X_{\sigma^2(u, v)},$$

where X_f denotes the Hamiltonian vector field generated by function f

$$4. d\sigma^2 + [\sigma^1, \sigma^2] = 0, \text{ i.e.}$$

$$\mathcal{L}_{\sigma^1(u)}\sigma^2(v, w) = \sigma^2([u, v], w) + \sigma^2(v, [u, w]).$$

Finally, let us see what happens to σ when we choose a different splitting of \mathfrak{g}_M to $\mathfrak{g} \oplus C^\infty(M)$. The resulting *gauge transformation* of σ by a $\tau \in \mathfrak{g}^* \otimes C^\infty(M)$ is explicitly:

1. $\sigma^0 \mapsto \sigma^0$, i.e. the Poisson structure on M doesn't change
2. $\sigma^1 \mapsto \sigma^1 + [\tau, \sigma^0]$, i.e. $\sigma^1(u) \mapsto \sigma^1(u) + X_{\tau(u)}$
3. $\sigma^2 \mapsto \sigma^2 + d\tau + [\tau, \sigma^1] + [\tau, [\tau, \sigma^0]]/2$, i.e.

$$\sigma^2(u, v) \mapsto \sigma^2(u, v) + \tau([u, v]) + \mathcal{L}_{\sigma^1(u)}\tau(v) - \mathcal{L}_{\sigma^1(v)}\tau(u) + \{\tau(u), \tau(v)\}/2$$

2. Up to Homotopy Actions as Poisson Principal Bundles

PROPOSITION 2.1. *The category of \mathfrak{g} -HPA's is equivalent to the category of Poisson principal \mathfrak{g}^* -bundles. Given a Poisson principal \mathfrak{g}^* -bundle $P \rightarrow M$, we construct the corresponding \mathfrak{g} -HPA on M as follows: the elements of \mathfrak{g}_M over $u \in \mathfrak{g}$ are functions f on P that change by $\langle u, \alpha \rangle$ under the action of $\alpha \in \mathfrak{g}^*$ on P ; the bracket on \mathfrak{g}_M is the Poisson bracket on P .*

Here \mathfrak{g}^* is a Poisson Lie group in the standard way (with the Kirillov–Kostant Poisson structure $\pi_{\mathfrak{g}^*}$); a Poisson principal \mathfrak{g}^* -bundle is a principal \mathfrak{g}^* -bundle $P \rightarrow M$ with a Poisson structure on P such that the action $P \times \mathfrak{g}^* \rightarrow P$ is a Poisson map.

Proof of the proposition is straightforward. If we choose a trivialization of P , in terms of the decomposition

$$\bigwedge^2 T(M \times \mathfrak{g}^*) = \bigwedge^2 TM \oplus TM \otimes \mathfrak{g}^* \oplus \bigwedge^2 \mathfrak{g}^*,$$

the Poisson structure on $P = M \times \mathfrak{g}^*$ is equal to

$$\sigma^0 + \sigma^1 + (\sigma^2 + \pi_{\mathfrak{g}^*}), \tag{3}$$

where $\pi_{\mathfrak{g}^*}$ is the Poisson structure on \mathfrak{g}^* and $\sigma = \sigma^0 + \sigma^1 + \sigma^2$ satisfies the MC equation (1), since $0 = [\sigma + \pi_{\mathfrak{g}^*}, \sigma + \pi_{\mathfrak{g}^*}] = 2[\pi_{\mathfrak{g}^*}, \sigma] + [\sigma, \sigma] = 2d\sigma + [\sigma, \sigma]$. Choosing a different trivialization corresponds to a gauge transformation of σ .

As we noticed in Sect. 1, a true Poisson \mathfrak{g} -action can be characterized as a \mathfrak{g} -HPA together with a morphism to the trivial \mathfrak{g} -HPA on a point. In other words, a Poisson \mathfrak{g} -action is the same as a principal Poisson \mathfrak{g}^* -bundle P together with a \mathfrak{g}^* -equivariant Poisson map $P \rightarrow \mathfrak{g}^*$.

Here is a somewhat exotic example of an up to homotopy Poisson action. As was observed by Vaintrob [8], if $A \rightarrow M$ is a vector bundle, the structure of a Lie algebroid on A is equivalent to a degree-1 vector field Q on the graded supermanifold $A[1]$ such that $Q^2 = 0$, and also to a degree -1 odd Poisson structure π on

$A^*[1]$; a bialgebroid structure is then both π and Q on $A^*[1]$ such that $\mathcal{L}_Q\pi=0$. Poisson actions up to homotopy appear in quasi-bialgebroids: A is a quasi-bialgebroid if we have a degree -1 odd Poisson structure π on $A^*[1]$ and an up to homotopy action of the 1-dimensional graded Lie algebra generated by Q ($[Q, Q]=0$, $\deg Q=1$) on $A^*[1]$. In other words, as was noticed in [5], it is a Poisson principal $\mathbb{R}[2]$ -bundle $P \rightarrow A^*[1]$ (where $\mathbb{R}[2]$ has zero Poisson structure).

3. Up to Homotopy Action of a Group on an Algebra

An *up to homotopy action of a Lie algebra \mathfrak{g} on an associative algebra A* (a \mathfrak{g} -HAA on A for short) is an extension \mathfrak{g}_A of \mathfrak{g} by A (where A is considered as a Lie algebra with the bracket given by the commutator), such that for any $X \in \mathfrak{g}_A$, the map $ad_X: A \rightarrow A$ is a derivation of the associative algebra A .

Similarly, an *up to homotopy action of a group G on an algebra* (a G -HAA) is a G -graded algebra A_G , i.e. an algebra of the form $A_G = \bigoplus_{g \in G} A_g$ with product mapping $A_g \otimes A_h$ to A_{gh} , such that every A_g contains an invertible element. The algebra A_e is the algebra on which G ‘acts up to homotopy’. In this definition, G is just an abstract group (and vector spaces are over an arbitrary field). In a smooth definition, G would be a Lie group and A_g ’s would form a smooth vector bundle over G .

One can give a more combinatorial definition. Let us choose an invertible element $\langle g \rangle$ in every A_g . For every $g \in G$ we have an automorphism $\rho(g)$ of A_e , given by $\rho(g)a = \langle g \rangle a \langle g \rangle^{-1}$. This ρ is not necessarily an action of G ; instead, we have

$$\rho(g)\rho(h)a = c(g, h)(\rho(gh)a)c(g, h)^{-1}, \quad (4)$$

where the elements $c(g, h) \in A_e$ are given by $c(g, h) = \langle g \rangle \langle h \rangle \langle gh \rangle^{-1}$. If we compute $\langle g_1 \rangle \langle g_2 \rangle \langle g_3 \rangle$ in two ways, we get a cocycle identity

$$c(g_1, g_2)c(g_1g_2, g_3) = (\rho(g_1)c(g_2, g_3))c(g_1, g_2g_3). \quad (5)$$

Vice versa, when we are given an algebra A_e , its automorphisms $\rho(g)$ and invertible elements $c(g, h)$ satisfying (4) and (5), we get an up to homotopy action.

This is indeed a non-commutative analogue of a principal Poisson \mathfrak{g}^* -bundle. To get the analogue one replaces Poisson spaces by associative algebras and reverses the arrows; the counterpart of \mathfrak{g}^* is a group algebra $\mathbb{F}[G]$ (where \mathbb{F} is our base field). The analogue of a Poisson space with a Poisson action of \mathfrak{g}^* is then an algebra with a compatible structure of a $\mathbb{F}[G]$ comodule, i.e. a G -graded algebra. The analogue of a principal \mathfrak{g}^* -action is the existence of invertible elements in every A_g ; the algebra A_e is the analogue of the base of the principal bundle.

A true action of G on an algebra A_e should be, of course, a special case of an up to homotopy action. All the spaces A_g are in this case equal to A_e ; to distinguish between them, let us denote A_g by $A_e \otimes g$ (its elements will be denoted by $a \otimes g$, $a \in A_e$). The product $A_g \times A_h \rightarrow A_{gh}$ is given by $(a \otimes g)(b \otimes h) = (a \cdot b) \otimes gh$. This G -graded algebra is the crossed product of A_e with G .

As an example of a different type, let \tilde{G} be a central extension of G by \mathbb{F}^* , where \mathbb{F} is our base field. Then the line bundle associated to the principal \mathbb{F}^* -bundle $\tilde{G} \rightarrow G$ is an up to homotopy action. And vice versa, every up to homotopy action which is a line bundle (i.e. where A_g 's are 1-dimensional), is of this type; in other words, an up to homotopy action of G on \mathbb{F} is equivalent to a central extension of G by \mathbb{F}^* .

4. Poisson versus Hamiltonian Actions: Crossed Products in Poisson Geometry

For any Poisson manifold M , a Poisson map $M \rightarrow \mathfrak{g}^*$ (a moment map) gives rise to a Poisson action of \mathfrak{g} on M ; Poisson actions of this form are called *Hamiltonian*. For every Poisson action one can construct a universal Hamiltonian action; more precisely, the forgetful functor

$$\text{Hamiltonian actions} \rightarrow \text{Poisson actions} \quad (6)$$

has a left adjoint. As we shall see, this adjoint is connected with up to homotopy Poisson actions.

As we noticed in Sect. 1, a Poisson action of \mathfrak{g} on M is the same as a \mathfrak{g} -HPA on M and a morphism to the trivial \mathfrak{g} -HPA on a point. Using the language of principal Poisson bundles, a Poisson action of \mathfrak{g} on M is thus the same as a principal Poisson \mathfrak{g}^* bundle $P \rightarrow M$ with a Poisson \mathfrak{g}^* -equivariant map $P \rightarrow \mathfrak{g}^*$. Since we have a Poisson map $P \rightarrow \mathfrak{g}^*$, on P we have a Hamiltonian action of \mathfrak{g} . Thus for any Poisson action (on M) we can find a Hamiltonian action (on $P = M \times \mathfrak{g}^*$); this operation is left adjoint to the forgetful functor (6), i.e. it satisfies a universal property:

PROPOSITION 4.1. *The forgetful functor (6) has a left adjoint given by $M \mapsto P = M \times \mathfrak{g}^*$, with the Poisson structure given by (3) (with σ^0 the Poisson structure on M , σ^1 the action of \mathfrak{g} on M and $\sigma^2 = 0$).*

Proof. of this proposition is straightforward: if N is a Poisson manifold with a moment map $\mu: N \rightarrow \mathfrak{g}^*$, and if $f: N \rightarrow M$ is a \mathfrak{g} -equivariant Poisson map, we should provide a unique Poisson map $\tilde{f}: N \rightarrow P$ commuting with the moment maps; the map \tilde{f} is simply $f \times \mu$. \square

Let us translate the previous to the world of noncommutative algebras. There the analogue of a moment map is a product and unit preserving map $\mu: G \rightarrow A$ (i.e. an algebra morphism $\mathbb{F}[G] \rightarrow A$); it gives an action of G on A by $g \cdot a = \mu(g)a\mu(g^{-1})$. The right-adjoint operation is the crossed product: given an action of G on A , it is $A \otimes \mathbb{F}[G]$ as a vector space, with the product given by

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = (a_1 g_1 \cdot a_2) \otimes (g_1 g_2);$$

the map μ is given by $\mu(g) = 1 \otimes g$.

We can also generalize our proposition to so called non-equivariant moment maps, i.e. Poisson maps $M \rightarrow \mathfrak{p}$, where \mathfrak{p} is a principal Poisson \mathfrak{g}^* -homogeneous space. Such a map again gives a Poisson action of \mathfrak{g} on M . The adjoint functor is the same as in the proposition: $M \mapsto M \times \mathfrak{p}$ (with the mixed part of the Poisson structure equal to the action of \mathfrak{g} on M). Notice that \mathfrak{p} is a \mathfrak{g} -HPA on a point. A \mathfrak{g}^* equivariant map $P \rightarrow \mathfrak{p}$ makes a principal Poisson \mathfrak{g}^* -bundle $P \rightarrow M$ to a true Poisson action of \mathfrak{g} on M .

The non-commutative analogue of this generalization uses a central extension \tilde{G} of a group G by \mathbb{F}^* . A non-equivariant moment map is a product preserving map $\mu: \tilde{G} \rightarrow A$ which is the identity on \mathbb{F}^* . Such a map gives an action of G on A via $g \cdot a = \mu(\tilde{g})a\mu(\tilde{g}^{-1})$, where $\tilde{g} \in \tilde{G}$ is any element over $g \in G$. There is an adjoint functor that is an obvious generalization of the crossed product.

5. Quantization of up to Homotopy Poisson Actions

The formality theorem of Kontsevich gives us a quantization of up to homotopy Poisson actions that are (in a sense specified below) formally close to true actions. The resulting up to homotopy action on the quantized algebra of functions will be also formally close to a true action.

Let thus σ^0 be a Poisson structure on M , formally depending on \hbar , such that $\sigma^0 = O(\hbar)$ (i.e. the kind of Poisson structure that is used in deformation quantization):

$$\sigma^0 = \sigma_{(1)}^0 \hbar + \sigma_{(2)}^0 \hbar^2 + \sigma_{(3)}^0 \hbar^3 + \dots$$

σ^1 and σ^2 also depend formally on \hbar , but are allowed to be $O(1)$. However, we demand $\sigma_{(0)}^1$ to be a true action of \mathfrak{g} on M . For the lack of fantasy we'll call such a HPA *formally good*.

Since $\sigma = \sigma^0 + \sigma^1 + \sigma^2$ satisfies the MC equation (1), it can be directly plugged into the formality theorem of Kontsevich, but only if $\sigma^1 = O(\hbar)$. If we want to allow $\sigma_{(0)}^1 \neq 0$, we need to use a rather straightforward trick. Under our assumptions we'll get a \mathfrak{g} -HAA on the algebra of quantized functions on M . If moreover the \mathfrak{g} -action $\sigma_{(0)}^1$ comes from an action of the simply-connected Lie group G on M , we'll also get a G -HAA.

We shall use quantization of Poisson families from [4] (and terminology from [6]). Recall that an *algebroid* over a set S is a linear category whose set of objects is S and where any two objects are isomorphic. In deformation quantization, algebroids are produced by quantization of *Hamiltonian families of formal Poisson structures*. Such a family is by definition a solution of the MC equation in the DGLA $\Omega(B) \hat{\otimes} L'_M$, where B and M are manifolds, $\hat{\otimes}$ is the completed tensor product (it just means that the coefficients are arbitrary smooth functions on $B \times M$) and the DGLA L'_M is given by

$$L_M'^{-1} = C^\infty(M)[[\hbar]]$$

$$L_M^i = \hbar \Gamma(\bigwedge^{i+1} TM)[[\hbar]].$$

If we choose a family of connections on M parametrized by B , we can quantize the Hamiltonian family to a “tight family of $*$ -products”, i.e. to a solution of the MC equation in $\Omega(B) \hat{\otimes} L_M''$, where the DGLA L_M'' is given by

$$L_M''^{-1} = C^\infty(M)[[\hbar]]$$

$$L_M''^i = \hbar PD^{i+1}[[\hbar]],$$

where PD^{i+1} is the space of polydifferential operators

$$\underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{i+1 \text{ times}} \rightarrow C^\infty(M).$$

As Kontsevich proved, this quantization is natural (i.e. diffeomorphism-invariant) and local. Moreover, if we suppose that B is 2-connected, the quantized family gives us naturally an algebroid over B .

Suppose now that G is a group and that we are given a G -HAA A_G . We then get an algebroid over G , defined by $\text{Hom}(g, h) = A_{g^{-1}h}$. Vice versa, given an algebroid over G , in which we can identify $\text{Hom}(g, h)$'s for a fixed $g^{-1}h$ in a way compatible with the product, we get an up to homotopy action.

We shall now take for G the 1-connected (and therefore also 2-connected) Lie group integrating \mathfrak{g} , set $B = G$, construct a Hamiltonian family, quantize it to an algebroid over G , and finally notice that the family is right G -invariant, which will thus turn the algebroid into a G -HAA.

The first thing is to notice the connection between Hamiltonian families of formal Poisson structures and formally good \mathfrak{g} -HPAs:

LEMMA 5.1. *Let ρ be a (left) action of \mathfrak{g} on M . A formally good \mathfrak{g} -HPA σ on M , such that $\sigma_{(0)}^1 = \rho$, is equivalent to a right- \mathfrak{g} -invariant Hamiltonian family of formal Poisson structures on M parametrized by G , where the right \mathfrak{g} action is by right translations on G and by $-\rho$ on M .*

Suppose now that the \mathfrak{g} -action ρ integrates to a G -action. Then we can choose a right- G -invariant family of connections on M parametrized by G (just choose one connection on M and then use the action). The algebroid over G we get will be right- G -invariant, so by the above remarks we get a G -HAA. We also get a \mathfrak{g} -HAA: after quantization we have a right- G -invariant solution of the MC equation in the DGLA $\Omega(G) \hat{\otimes} L_M''$.

If ρ doesn't integrate to a G -action, we can choose the invariant family of connections only locally. It is, of course, sufficient to get a \mathfrak{g} -HAA. We also get a “local G -HAA”, though we leave it to the reader to formulate the definition.

Let us summarize the outcome of this section:

PROPOSITION 5.2. *Let ρ be a (left) action of \mathfrak{g} on M and let σ be a formally good \mathfrak{g} -HPA on M with $\sigma_{(0)}^1 = \rho$.*

1. *There is a quantization of σ to a \mathfrak{g} -HAA on a quantized algebra of functions on M*
2. *If moreover ρ integrates to an action of G (the 1-connected group integrating \mathfrak{g}) then we also get a G -HAA.*

6. Generalizations for Lie bialgebras and Hopf Algebras; Open Ends

There is now an obvious definition of up to homotopy actions of Lie bialgebras. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra; let us also choose a Poisson Lie group G^* integrating \mathfrak{g}^* . An *up to homotopy Poisson action* of (\mathfrak{g}, G^*) ((\mathfrak{g}, G^*) -HPA for short) is, by definition, a principal Poisson G^* bundle. Such a bundle $P \rightarrow M$ with a G^* -equivariant Poisson map $P \rightarrow G^*$ is again equivalent to a Poisson action of $(\mathfrak{g}, \mathfrak{g}^*)$ on M : indeed, P becomes $M \times G^*$, and the mixed term of the Poisson structure is the action of \mathfrak{g} . Moreover, this operation is left-adjoint to the forgetful functor

$$\text{Poisson maps to } G^* \text{ (moment maps)} \rightarrow \text{Poisson actions of } (\mathfrak{g}, \mathfrak{g}^*).$$

More generally, we have the following:

PROPOSITION 6.1. *Let \mathfrak{P} be a right principal Poisson homogeneous space of G^* . If M is a Poisson manifold, a Poisson map $\mu : M \rightarrow \mathfrak{P}$ generates a (left) Poisson action ρ of $(\mathfrak{g}, \mathfrak{g}^*)$ on M via $\rho(v)_x = \langle \pi_M, \mu^* v \rangle$, where $x \in M$, $v \in \mathfrak{g} \cong T_{\mu(x)}^* \mathfrak{P}$ (these spaces are isomorphic because \mathfrak{P} is a principal G^* space) and π_M is the Poisson structure on M .¹ This functor has a left adjoint, given by $M \mapsto P = M \times \mathfrak{P}$, with the Poisson structure $\pi_P = \pi_M + \pi_{\mathfrak{P}}$ + the \mathfrak{g} -action. Any Poisson principal G^* -bundle $P \rightarrow M$ with a Poisson equivariant map $P \rightarrow \mathfrak{P}$ is of this form.*

Let \mathfrak{d} be the Drinfeld double of $(\mathfrak{g}, \mathfrak{g}^*)$. Poisson principal homogeneous G^* -spaces, or in other words Poisson principal G^* -bundles over a point, are classified by Lagrangian subalgebras of \mathfrak{d} transversal to \mathfrak{g}^* [2]. There is a similar classification of general Poisson principal bundles. In the case of the trivial bundle we have:

PROPOSITION 6.2. *A Poisson structure on $P = M \times G^*$, making it to a Poisson principal G^* bundle, is the same as a Dirac structure in the Courant algebroid $\mathfrak{d} \oplus (T \oplus T^*)M$, transversal to $\mathfrak{g}^* \oplus TM$.*

If we also require the projection $M \times G^* \rightarrow G^*$ to be Poisson (i.e. when we are interested in the true Poisson actions of $(\mathfrak{g}, \mathfrak{g}^*)$ on M), the Dirac structure in

¹Up to this point the proposition is of course well known and is included just for completeness.

$\mathfrak{d} \oplus (T \oplus T^*)M$ has to project to $\mathfrak{g} \subset \mathfrak{d}$ under the projection $\mathfrak{d} \oplus (T \oplus T^*)M \rightarrow \mathfrak{d}$. This is a special case of an observation from [1]: if $\mathfrak{g} \subset \mathfrak{d}$ is a Lie quasi-bialgebra, a $\mathfrak{g} \subset \mathfrak{d}$ -quasi-Poisson structure on a manifold M (i.e. a “quasi-Poisson action of \mathfrak{g} on M ”) is the same as a Dirac structure in $\mathfrak{d} \oplus (T \oplus T^*)M$ that projects to $\mathfrak{g} \subset \mathfrak{d}$ under the projection $\mathfrak{d} \oplus (T \oplus T^*)M \rightarrow \mathfrak{d}$, and that is transversal to TM . (This suggests a definition of “up to homotopy quasi-Poisson structures”, but it’s not clear what it would be good for.)

In the case of a general (non-trivial) principal G^* -bundle $P \rightarrow M$, the Poisson structures on P making it to a Poisson principal bundle, are again the same as Dirac structures in some transitive Courant algebroid $C \rightarrow M$, transversal to $A \subset C$, where A is the Atiyah Lie algebroid corresponding to P . The Courant algebroid C is obtained by first inducing P to a principal D -bundle (i.e. by creating the associated principal D -bundle to P), where D is a Lie group integrating \mathfrak{d} , and then using the reduction procedure from [7]. (Since the reduction procedure is shown in [7] to produce basically all transitive Courant algebroids, this again suggests some “up to homotopy and quasi” directions). The fact that we get a Lie bialgebroid (a pair of transversal Dirac structures in a Courant algebroid) is not surprising – the corresponding Poisson groupoid is the gauge groupoid of P .

Homotopy Poisson actions, as we defined them, involve a Lie algebra \mathfrak{g} . To integrate a HPA to a kind of “action” of a Lie group G , we can take the Poisson principal bundle P , form its gauge groupoid (that is automatically a Poisson groupoid), and finally pass to its dual Poisson groupoid $\tilde{\Gamma}$ if it exists. The Poisson groupoid $\tilde{\Gamma}$ can be characterized as follows: it is an extension of G by a symplectic groupoid Γ integrating the Poisson manifold M . This kind of extension might be called “up to homotopy action of G on M ”; notice that it requires M to be integrable. It is a direct translation of the definition of G -HAAs from Sect. 3.

Let us now switch to the world of non-commutative algebras, replacing Lie bialgebras by Hopf algebras. Let H be a Hopf algebra (the case of $H = \mathbb{F}[G]$ will lead back to G -HAA’s). The substitute for an action of a group G up to homotopy on an algebra A is a H -Galois extension of A , i.e. an algebra B , with a H -comodule structure $c: B \rightarrow B \otimes H$, $b \mapsto b_{(1)} \otimes b_{(2)}$ that is also an algebra map, and an algebra map $i: A \rightarrow B$, such that

$$i(A) = \{b \in B \mid c(b) = b \otimes 1\}$$

and such that the following map is an isomorphism:

$$B \otimes_A B \rightarrow B \otimes H, \quad x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}.$$

I don’t know if there is a theorem that would give a quantization of a (\mathfrak{g}, G^*) -HPA on M to a Hopf–Galois extension of a quantized algebra of functions on M .

The non-commutative analogue of Proposition 6.1 remains valid. The analogue of \mathfrak{P} is an H -Galois extensions of the base field \mathbb{F} ; in the case of $H = \mathbb{F}[G]$ this is the same as a central extension of G by \mathbb{F}^* .

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